First Welfare Theorem

Econ 3030 Fall 2025

Lecture 20

Outline

1. First Welfare Theorem
2. Preliminaries to Second Welfare Theorem

Past Definitions

A feasible allocation $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ *is* Pareto optimal *if there is no other feasible allocation* (x, y) *such that*

$$\mathbf{x}_i \succsim_i \hat{\mathbf{x}}_i \quad \text{for all } i \quad \text{and} \quad \mathbf{x}_i \succ_i \hat{\mathbf{x}}_i \quad \text{for some } i.$$

An allocation $(\mathbf{x}^*, \mathbf{y}^*)$ *and a price vector* $\mathbf{p}^* \in \mathbb{R}^L_+$ *form a* competitive equilibrium *if*

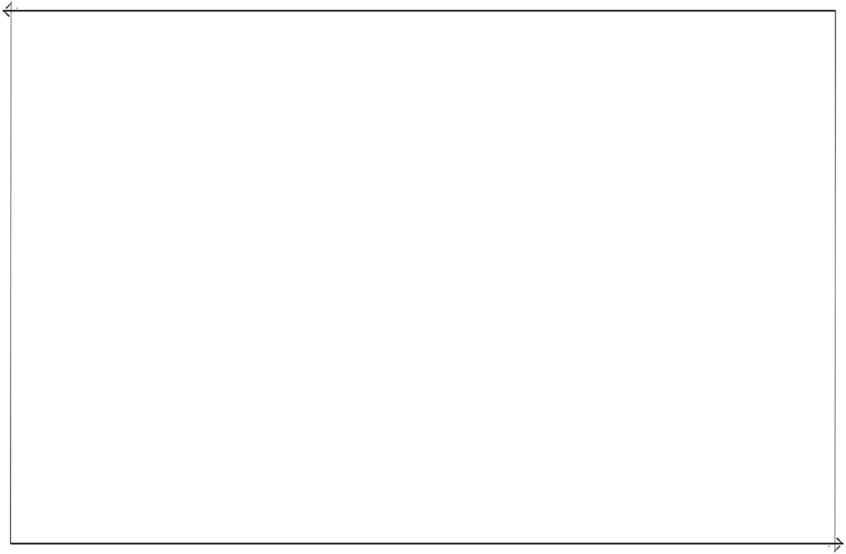
1. *for* $j = 1, ..., J$: $\mathbf{p}^* \cdot \mathbf{y}_j \leq \mathbf{p}^* \cdot \mathbf{y}_j^*$ *for all* $\mathbf{y}_j \in Y_j$
2. *for* $i = 1, ..., I$: $\mathbf{x}_i^* \succsim_i \mathbf{x}_i$ *for all* $\mathbf{x}_i \in \{\mathbf{x}_i \in X_i : \mathbf{p}^* \cdot \mathbf{x}_i \leq \mathbf{p}^* \cdot \boldsymbol{\omega}_i + \sum_j \theta_{ij}(\mathbf{p}^* \cdot \mathbf{y}_j^*)\}$
3. $\sum_i \mathbf{x}_i^* \leq \sum_i \boldsymbol{\omega}_i + \sum_j \mathbf{y}_j^*$ *and if* $\sum_{i=1}^I x_{li}^* < \sum_{i=1}^I \omega_{li} + \sum_{j=1}^J y_{lj}^*$ *then* $p_l^* = 0$.

Relationship between competitive equilibrium and Pareto efficiency

- Is any competitive equilibirum Pareto efficient? First Welfare Theorem.
  - This is about ruling out allocations that can Pareto dominate the equilibrium allocation.
- Is any Pareto efficient allocation (part of) a competitive equilibrium? Second Welfare Theorem.
  - This is about finding prices that make the efficient allocation an equilibirum.

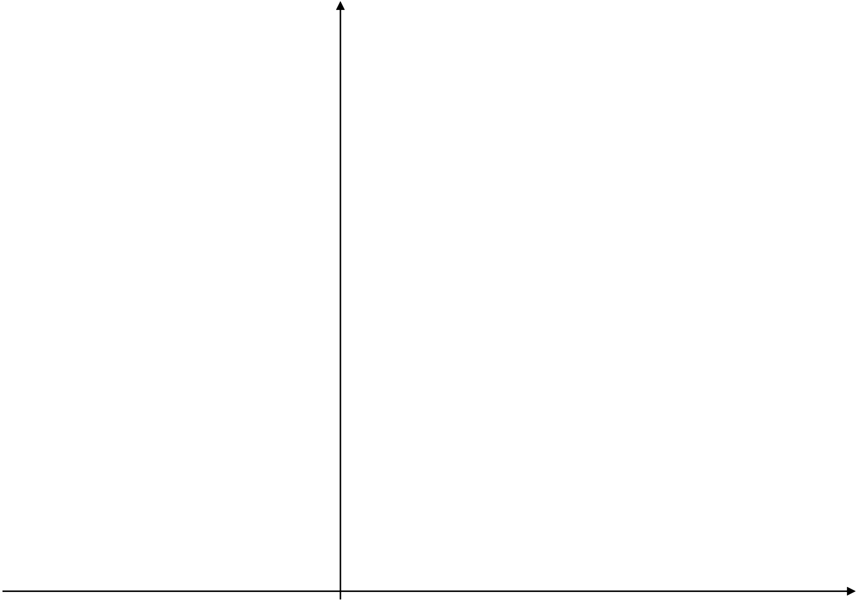
First Welfare Theorem: Edgeworth Box

Things seem easy



First Welfare Theorem: Representative Agent

Things seem easy



First Welfare Theorem: Counterexample

An Edgeworth Box Economy: two-person, two-good exchange economy

- Consumers a and b utility functions are

$$U_a(x_{1a}, x_{2a}) = 7 \qquad \text{and} \qquad U_b(x_{1b}, x_{2b}) = x_{1b}x_{2b}$$

  while the initial endowments are $\boldsymbol{\omega}_a = (2, 0) \qquad \text{and} \qquad \boldsymbol{\omega}_b = (0, 2)$

- CLAIM: $\mathbf{x}_a^* = (1, 1)$, $\mathbf{x}_b^* = (1, 1)$, and $\mathbf{p}^* = (1, 1)$ form a competitive equilibirum.
  - a's utility is maximized at $\mathbf{x}_a^*$.
  - b's utility when her income equals 2 is maximized at $\mathbf{x}_b^*$ (this is a Cobb-Douglas utility function with equal exponents, so spending half her income on each good is optimal).
  - $\mathbf{x}_a^* + \mathbf{x}_b^* = (2, 2) = \boldsymbol{\omega}_a + \boldsymbol{\omega}_b$ so demand equals supply.
- Is this allocation Pareto optimal? No:
  - $\hat{\mathbf{x}}_a = (0, 0)$ and $\hat{\mathbf{x}}_b = (2, 2)$ Pareto dominates $\mathbf{x}_a^*$, $\mathbf{x}_b^*$ since consumer a has the same utility while consumer b's utility is higher.

- How do we rule examples like this out?
- Need consumers preferences to be locally non satiated (there is always something nearby that makes the consumer better off).

Local Non Satiation

Definition

A preference ordering $\succsim_i$ on X_i is satiated at $\mathbf{y}$ if there exists no $\mathbf{x}$ in X_i such that $\mathbf{x} \succ_i \mathbf{y}$.

Definition

The preference relation $\succsim_i$ on X_i is locally non-satiated if for every $\mathbf{x}$ in X_i and for every $\varepsilon > 0$ there exists an $\mathbf{x}'$ in X_i such that $\|\mathbf{x}' - \mathbf{x}\| < \varepsilon$ and $\mathbf{x}' \succ_i \mathbf{x}$.

- Remember: $\|\mathbf{y} - \mathbf{z}\| = \sqrt{\sum_{l=1}^{L} (y_l - z_l)^2}$ is the Euclidean distance between two points.

Remark

If $\succsim_i$ is continuous and locally non-satiated it is represented by a locally non-satiated utility function; then, any closed consumption set must be unbounded (or there would be a global satiation point).

Local Non Satiation and Walrasian Demand

Lemma

Suppose $\succsim_i$ *is locally non-satiated, and let* x_i^* *be defined as:*

$$\mathbf{x}_i^* \succsim_i \mathbf{x}_i \qquad \text{for all } \mathbf{x}_i \in \{\mathbf{x}_i \in X_i : \mathbf{p} \cdot \mathbf{x}_i \leq w_i\}.$$

Then

$$\mathbf{x}_i \succsim_i \mathbf{x}_i^* \qquad \text{implies} \qquad \mathbf{p} \cdot \mathbf{x}_i \geq w_i$$

and

$$\mathbf{x}_i \succ_i \mathbf{x}_i^* \qquad \text{implies} \qquad \mathbf{p} \cdot \mathbf{x}_i > w_i$$

- If a consumption vector is weakly preferred to a maximal consumption bundle (i.e. an element of the Walrasian demand correspondence), it cannot cost strictly less.
- If a consumption vector is strictly preferred to a maximal bundle, it must not be affordable
  - If not the consumer would have chosen it and been better-off.
- The formal proof is for the Problem Set. (HINT: Draw a picture before starting the proof).

First Welfare Theorem

Theorem (First Fundamental Theorem of Welfare Economics)

Suppose each consumer's preferences are locally non-satiated. If $\mathbf{x}^*, \mathbf{y}^*$ *and prices* $\mathbf{p}^*$ *form a competitive equilibrium, then* $\mathbf{x}^*, \mathbf{y}^*$ *is Pareto optimal.*

- Equilibrium prices plus individuals' maximization yield a Pareto efficient allocation.
  - The planner cannot improve an equilibrium allocation.
- The theorem makes mild assumptions on individuals' preference relations.
- Local non-satiation has bite: there is always a more desirable commodity bundle nearby.
- There is another assumption implicit in our framework: lack of externalities (more later).

Proof of the First Welfare Theorem

By contradiction:

Suppose not: there exists a feasible allocation $\mathbf{x}, \mathbf{y}$ *such such that* $\mathbf{x}_i \succsim_i \mathbf{x}_i^*$ *for all* i*, and* $\mathbf{x}_i \succ_i \mathbf{x}_i^*$ *for some* i*.*

- *By local non satiation,* $\begin{array}{lll} \mathbf{x}_i \succsim_i \mathbf{x}_i^* & \textit{implies} & \mathbf{p}^* \cdot \mathbf{x}_i \geq \mathbf{p}^* \cdot \boldsymbol{\omega}_i + \sum_j \theta_{ij}(\mathbf{p}^* \cdot \mathbf{y}_j^*) \\ \mathbf{x}_i \succ_i \mathbf{x}_i^* & \textit{implies} & \mathbf{p}^* \cdot \mathbf{x}_i > \mathbf{p}^* \cdot \boldsymbol{\omega}_i + \sum_j \theta_{ij}(\mathbf{p}^* \cdot \mathbf{y}_j^*) \end{array}$
- *Therefore, summing over consumers*

$$\sum_{i=1}^{I} \mathbf{p}^* \cdot \mathbf{x}_i > \sum_{i=1}^{I} \mathbf{p}^* \cdot \boldsymbol{\omega}_i + \sum_{i=1}^{I}\sum_{j=1}^{J} \theta_{ij}(\mathbf{p}^* \cdot \mathbf{y}_j^*) \overset{accounting}{=} \sum_{i=1}^{I} \mathbf{p}^* \cdot \boldsymbol{\omega}_i + \sum_{j=1}^{J} \mathbf{p}^* \cdot \mathbf{y}_j^*$$

- *Since each* $\mathbf{y}_j^*$ *maximizes profits at prices* $\mathbf{p}^*$*, we also have* $\sum_{j=1}^{J} \mathbf{p}^* \cdot \mathbf{y}_j^* \geq \sum_{j=1}^{J} \mathbf{p}^* \cdot \mathbf{y}_j$
- *Substituting this into the previous inequality:*

$$\sum_{i=1}^{I} \mathbf{p}^* \cdot \mathbf{x}_i > \sum_{i=1}^{I} \mathbf{p}^* \cdot \boldsymbol{\omega}_i + \sum_{j=1}^{J} \mathbf{p}^* \cdot \mathbf{y}_j$$

Proof of the First Welfare Theorem (continued)

We have shown that

$$\sum_{i=1}^{I} \mathbf{p}^* \cdot \mathbf{x}_i > \sum_{i=1}^{I} \mathbf{p}^* \cdot \boldsymbol{\omega}_i + \sum_{j=1}^{J} \mathbf{p}^* \cdot \mathbf{y}_j$$

- *This contradicts feasibility of* $(\mathbf{x}, \mathbf{y})$ *because*

$$\sum_{i=1}^{I} x_{li} \leq \sum_{i=1}^{I} \omega_{li} + \sum_{j=1}^{J} y_{lj} \qquad \Rightarrow \qquad \sum_{i=1}^{I} p_l^* x_{li} \leq \sum_{i=1}^{I} p_l^* \omega_{li} + \sum_{j=1}^{J} p_l^* y_{lj}$$

- *Summing over goods:*

$$\sum_{l=1}^{L} \sum_{i=1}^{I} p_l^* x_{li} \leq \sum_{l=1}^{L} \sum_{i=1}^{I} p_l^* \omega_{li} + \sum_{l=1}^{L} \sum_{j=1}^{J} p_l^* y_{lj}$$

- *which implies*

$$\sum_{i=1}^{I} \mathbf{p}^* \cdot \mathbf{x}_i \leq \sum_{i=1}^{I} \mathbf{p}^* \cdot \boldsymbol{\omega}_i + \sum_{j=1}^{J} \mathbf{p}^* \cdot \mathbf{y}_j$$

□

First Welfare Theorem

Theorem (First Fundamental Theorem of Welfare Economics)

Suppose each consumer's preferences are locally non-satiated. If x^*, y^* and prices p^* form a competitive equilibrium, then x^*, y^* is Pareto optimal.

- The theorem says that as far as Pareto optimality goes the social planner cannot improve upon a competitive equilibrium.

Remark

The theorem needs only a seemingly weak assumption to obtain a pretty strong conclusion.

- On the other hand, the important assumption of absence of externalities is implicit in the way we set up the theory.
- An externality is present when preferences or profit depend on more than one's own choices.

Externalities: An Example

An Edgeworth Box Economy (two goods and two consumers)

- Consumer B: $u_B(x_{1B}, x_{2B}) = x_{1B}x_{2B}$ and $\boldsymbol{\omega}_B = (0,2)$.
- Consumer A: $u_A(x_{1A}, x_{2A}, x_{1B}) = x_{1A}x_{2A} - x_{1B}$ and $\boldsymbol{\omega}_A = (2,0)$.
  - A suffers from B's consumption of the first good.
- CLAIM: $\mathbf{x}_A^* = (1,1)$, $\mathbf{x}_B^* = (1,1)$, and $\mathbf{p}^* = (1,1)$ form a competitive equilibirum.
  - Since A cannot choose x_{1B}, this is a constant in her utility function. Thus, A's utility is maximized by $\mathbf{x}_A^*$ at prices $\mathbf{p}^*$ (this is a Cobb-Douglas utility function with equal exponents, so spending half income on each good is optimal).
  - B's utility when her income equals 2 is maximized (this is a Cobb-Douglas utility function with equal exponents, so spending half her income on each good is optimal).
  - $\mathbf{x}_A^* + \mathbf{x}_B^* = (2,2) = \boldsymbol{\omega}_A + \boldsymbol{\omega}_B$.
- Is $(\mathbf{x}_A^*, \mathbf{x}_B^*)$ Pareto optimal? No: $(\hat{\mathbf{x}}_A, \hat{\mathbf{x}}_B) = ((\frac{5}{4}, \frac{2}{3}), (\frac{3}{4}, \frac{4}{3}))$ is a feasible Pareto improvement:

$$\begin{aligned} U_A(\hat{x}_{1A}, \hat{x}_{2A}, \hat{x}_{1B}) &= \frac{5}{4}\frac{2}{3} - \frac{3}{4} = \frac{1}{12} > U_A(x_{1A}^*, x_{2A}^*, x_{1B}^*) = 1 - 1 = 0 \\ U_B(\hat{x}_{1B}, \hat{x}_{2B}) &= \frac{3}{4}\frac{4}{3} = 1 = U_B(x_{1B}^*, x_{2B}^*) \end{aligned}$$

First Welfare Theorem: Externalities

- In the previous example, the first welfare theorem fails because A's utility depends on B's consumption.
- That is an example of a (negative) consumption externality: the more B consumes of the good, the worse-off A becomes.
- There can be also externalities in production.
- Externalities can also be positive.

Remark

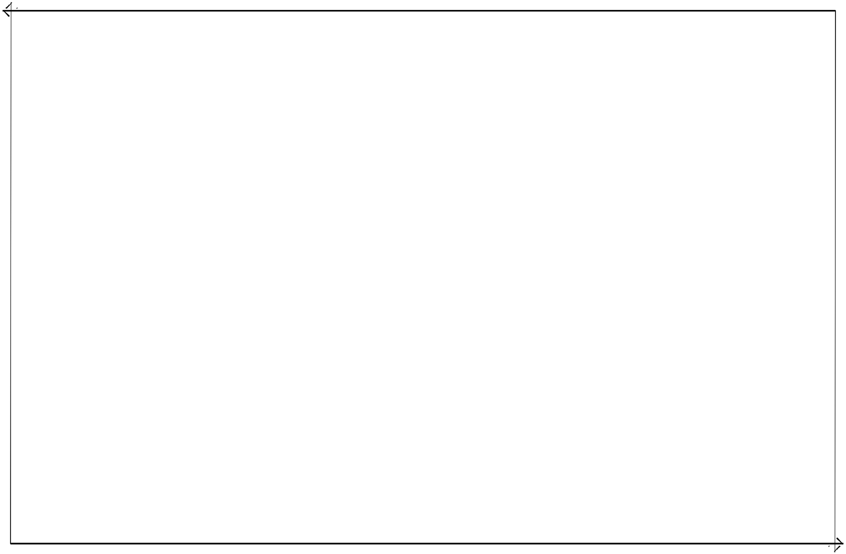
Among the assumptions implicit in our definition of preferences and production possibility sets, one is crucial for the first welfare theorem: there are no externalities in consumption or production.

Second Welfare Theorem: Preliminaries

- Next, we focus on a converse to the First Welfare Theorem.
- The statement will go something like this: under *some conditions*, any Pareto optimal allocation is part of a competitive equilibrium.
  - Today, we try to understand what these conditions must be. We will state and prove the theorem next class.
- Since an equilibrium must specify an allocation **and** prices, in order to prove that a Pareto optimal allocation is part of an equilibrium one needs to find the price vector that 'works' for that allocation.
- First, we see an obvious sense in which this cannot be done: Pareto optimality disregards the budget constraints.
  - This is fixed by appropriately adjusting the definition of equilibrium.
- Then, we see two counterexamples that stress the need for convexities.
  - These are fixed by assuming production sets and better-than sets are convex.
- Finally, we see an example showing that boundary issues can pose problems.
  - This is fixed by, again, adjusting the definition of equilibrium.

Second Welfare Theorem: Need Transfers

Example



Equilibrium With Transfers

Definition

Given an economy $\left(\{X_i, \succsim_i, \omega_i\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}\right)$, an allocation x^*, y^* and a price vector p^* constitute a price equilibrium with transfers if there exists a vector of wealth levels

$$\mathbf{w} = (w_1, w_2, ..., w_I) \qquad \text{with} \qquad \sum_{i=1}^{I} w_i = \sum_{i=1}^{I} \mathbf{p}^* \cdot \boldsymbol{\omega}_i + \sum_{j=1}^{J} \mathbf{p}^* \cdot \mathbf{y}_j^*$$

such that:

1. For each $j = 1, ..., J$: $\quad \mathbf{p}^* \cdot \mathbf{y}_j \leq \mathbf{p}^* \cdot \mathbf{y}_j^* \quad$ for all $y_j \in Y_j$
2. For each $i = 1, ..., I$: $\quad \mathbf{x}_i^* \succsim_i \mathbf{x}_i \quad$ for all $\mathbf{x}_i \in \{\mathbf{x}_i \in X_i : \mathbf{p}^* \cdot \mathbf{x}_i \leq w_i\}$
3. For each $l = 1, ..., L$: $\quad \sum_{i=1}^{I} x_{li}^* \leq \sum_{i=1}^{I} \omega_{li} + \sum_{j=1}^{J} y_{lj}^*, \quad$ and $p_l = 0$ if strict inequality

- Aggregate wealth is divided so that the consumers' budget constraints are satisfied.
- How is each consumer effected? They get a positive or negative transfer.
- A competitive equilibrium satisfies this: let $w_i = \mathbf{p}^* \cdot \boldsymbol{\omega}_i + \sum_{j=1}^{J} \theta_{ij}(\mathbf{p}^* \cdot \mathbf{y}_j^*)$.

Equilibrium With Transfers

Remark

The income transfers (across consumers) that achieve the budget levels in the previous definition are:

$$T_i = w_i - \left[\mathbf{p}^* \cdot \boldsymbol{\omega}_i + \sum_{j=1}^{J} \theta_{ij} \left(\mathbf{p}^* \cdot \mathbf{y}_j^* \right) \right]$$

- Summing over consumers, we get

$$\begin{aligned} \sum_{i=1}^{I} T_i &= \sum_{i=1}^{I} w_i - \left[\sum_{i=1}^{I} \mathbf{p}^* \cdot \boldsymbol{\omega}_i + \sum_{i=1}^{I} \sum_{j=1}^{J} \theta_{ij} \left(\mathbf{p}^* \cdot \mathbf{y}_j^* \right) \right] \\ &= \sum_{i=1}^{I} w_i - \left[\sum_{i=1}^{I} \mathbf{p}^* \cdot \boldsymbol{\omega}_i + \sum_{j=1}^{J} \mathbf{p}^* \cdot \mathbf{y}_j^* \right] \\ &= 0 \end{aligned}$$

- Transfers redistribute income so that the ‘aggregate budget’ balances.

Next Class

- Proof of the Second Fundamental Theorem of Welfare Economics.